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AC susceptibilities in spin glasses and dynamics in an ultrametric space

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Abstract. We show that the temperature and frequency dependence of the AC susceptibilities $\chi(\omega)$ in the 2D and 3D $\pm J$ Ising spin glasses (SGs) can be reproduced by a model based on the picture of motion in phase space as thermally activated hopping in an ultrametric space. The characteristic difference in the frequency dependence of $\mathrm{Im}\,\chi(\omega)$ in the 2D and 3D SGs, previously found by the Monte Carlo (MC) simulation, is reduced to a difference in a scaling of free-energy barriers with ultrametric distance. One assumption is needed to combine dynamics in an ultrametric space with a correlation function of the magnetization in the SGs. $\chi(\omega)$-values that were obtained previously by the MC simulation are reproduced based on our assumption with better results than those of Sibani. Further the relation obtained by the Lundgren *et al* $\mathrm{Im}\,\chi = -(\pi/2)[\mathrm{d}(\mathrm{Re}\,\chi)/\mathrm{d}(\ln\omega)]$ holds without taking a special limit $z \to 1$, where z is a branching ratio of the hierarchical structure.

1. Introduction

Recently AC susceptibilities $\chi(\omega)$ and relaxation time distribution $g(\tau)$ in the $\pm J$ Ising spin glasses (SGs) were calculated by Monte Carlo (MC) simulations, and characteristic differences were found in the two-dimensional (2D) and three-dimensional (3D) systems by Suzuki, Shirakura and Inawashiro (SSI) (1991). In this paper we intend to reproduce those behaviours by a model based on the picture of motion in phase space as thermally activated hopping in an ultrametric space.

Since Mezard *et al* (1984) showed that many pure states in the mean-field SG (Sherrington and Kirkpatrick 1975, Parisi 1980) are separated by free-energy barriers defining an ultrametric topology, much research has been devoted to dynamics in an ultrametric space (Ogielski and Stein 1985, Bachas and Huberman 1987). However, there have been only a few investigations which make detailed comparisons between SG dynamics and the dynamics in an ultrametric space. Sibani (1987) intended to reproduce the temperature and frequency dependence of the AC susceptibility $\chi(\omega)$ in a SG using the dynamics in an ultrametric space. Although he stated that his results and many experiments are in qualitatively good agreement, we suppose that there are many disagreements between them, even for qualitative behaviour. For example, the peak height of the imaginary part $\mathrm{Im}\,\chi(\omega)$ of the AC susceptibility given by Sibani increases with decreasing frequency ω. On the other hand, in many experiments on SGs with a finite transition temperature (Gunnarsson *et al* 1988) and in the MC simulation of the 3D $\pm J$ Ising SG by SSI, the peak height of $\mathrm{Im}\,\chi(\omega)$ decreases with decreasing ω. Also, as indicated by Sibani himself, the magnitude of $[\mathrm{Im}\,\chi(\omega)]/\mathrm{Re}\,\chi(\omega)]$ in his results is larger

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than that in many experimental results on SGs. Further the relation $\mathrm{Im}\,\chi(\omega) = (-\pi/2)\{\mathrm{d}[\mathrm{Re}\,\chi(\omega)]/\mathrm{d}(\ln\omega)\}$ obtained by Lundgren, Svedlindh and Beckman (LSB) (1981) and observed in many experiments on SGs does not hold in his model, except for a special limit $z \to 1$, where z is a branching ratio of the hierarchical structure.

We think that an assumption made by Sibani (1987) in order to combine the dynamics in the ultrametric space with the correlation function $C(t)$ of the magnetization in SGs is not appropriate. In this paper we present another assumption. On the basis of this assumption and the model presented by Ogielski and Stein (OS) (1985) with a fixed value of $z = 2$ in the ultrametric space, we show that $\chi(\omega)$- and $g(\tau)$-values in the 2D and 3D $\pm J$ Ising SGs can be reproduced, including even the difference in their characteristic features.

In section 2 we briefly describe the OS model and present one assumption in order to combine the dynamics with $C(t)$ in SGs. Then we derive expressions for $\chi(\omega)$ and $g(\tau)$. In section 3 these are calculated numerically and estimated analytically in three different cases, and the results are shown to resemble those of the experiments and the MC simulation. Section 4 contains a discussion.

2. Model and dynamical physical quantities

First we describe the OS model defined in an ultrametric space. The notation used in this paper is the same as in the work by OS and by SSI.

We consider the hierarchical structure of a branching ratio $z = 2$ (see figure 1 in the paper by OS). The number of hierarchical levels is assumed to be $n + 1$. The 2^n points on the top level are numbered as $0, 1, 2, \ldots, 2^n - 1$ and considered to correspond to metastable states in the high-temperature phase of the SG. The ultrametric distance d between two points is given by the number of branches that one must descend from the top level before the branches merge. Two points with the ultrametric distance d are supposed to be separated by the free-energy barrier Δ_d which is ranked so that $\Delta_1 < \Delta_2 < \ldots < \Delta_n$.

This problem is formulated as a random walk on the top level points. Let the probability of the particle being found at site i at time t be given by $P_i(t)$; hence, $\sum_{i=0}^{2^n-1} P_i(t) = 1$. Further, let the probability per unit time of jumping an ultrametric distance of d be given by ε_d, $d = 1, 2, \ldots, n$. OS obtained a solution for $P_i(t)$ with the initial condition $P_0(0) = 1$, $P_i(0) = 0$ $(i \geqslant 1)$, and the average distance travelled in time t given by

$$\langle d(t)\rangle \equiv \sum_{k=0}^{2^n-1} d(k,0)P_k(t) = n - 1 + 2^{-n} - \sum_{m=1}^{n} (1 - 2^{-m}) \exp(-\lambda_m t) \tag{1}$$

where

$$\lambda_m = a_m + \sum_{k=m}^{n} a_k \qquad a_k \equiv 2^{k-1}\varepsilon_k \tag{2}$$

and $d(k, j)$ is the ultametric distance between site k and site j. Because $a_k = 2^{k-1}\varepsilon_k$ represents the probability per unit time of jumping to any of the 2^{k-1} sites a distance k from a given starting site, we assume that $a_k = \exp(-\Delta_k/T)$ at temperature T.

Next we combine the above results with dynamical physical quantities in a SG. We assume that the correlation function $C(t) = [\langle M(0)M(t)\rangle]_{\mathrm{av}}$ of the magnetization in a SG decreases in proportion to the average distance $\langle d(t)\rangle$, i.e.

$$C(t)/N = 1 - \langle d(t)\rangle/(n - 1 + 2^{-n}) \tag{3}$$

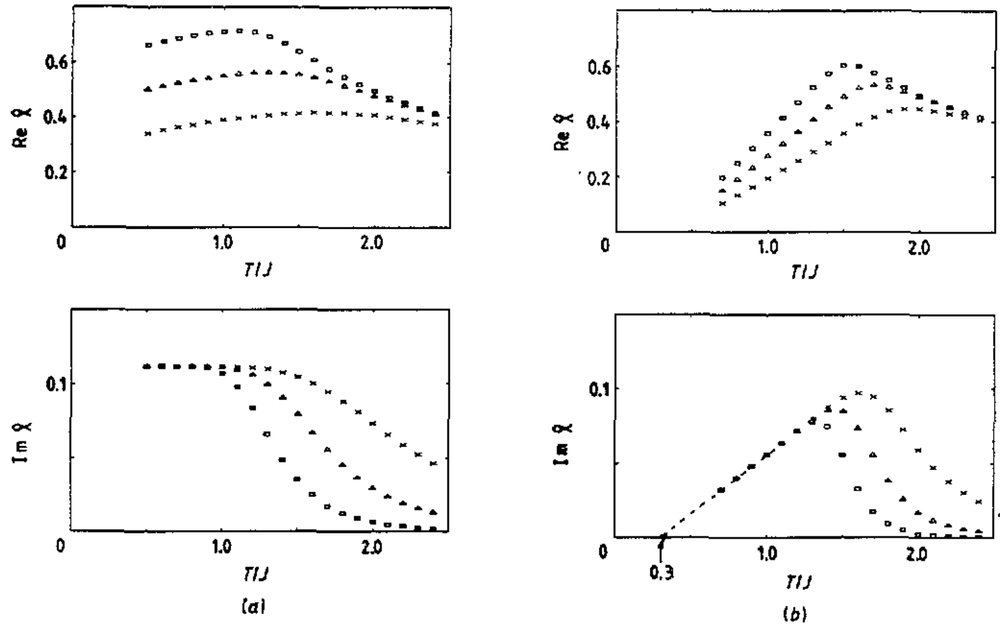

Figure 1. Re $\hat{\chi}$ and Im $\hat{\chi}$ versus T/J for case I with $\Delta/J = 0.014$ and $\omega = 2\pi/20$ ($\times$), $2\pi/200$ ($\triangle$) and $2\pi/2000$ ($\square$): (a) $n = 1000$; (b) $n = 1400J/(T - T_c)$ where $T_c/J = 0.3$.

where N is a number of spins and the factor $n - 1 + 2^{-n}$ is determined to ensure $\lim_{t\to\infty} C(t) = 0$. From (1) and (3), we obtain

$$\frac{C(t)}{N} = \sum_{m=1}^{n} \frac{(1 - 2^{-m}) \exp(-\lambda_m t)}{n - 1 + 2^{-n}}. \tag{4}$$

A relaxation time distribution $g(\tau)$ is defined by

$$\frac{C(t)}{N} = \int_0^{\infty} \mathrm{d}\tau\, g(\tau) \exp\left(-\frac{|t|}{\tau}\right). \tag{5}$$

Comparing (5) with (4), $g(\tau)$ is given by

$$g(\tau) = \sum_{m=1}^{n} \frac{(1 - 2^{-m})}{(n - 1 + 2^{-n})}\, \delta\left(\tau - \frac{1}{\lambda_m}\right). \tag{6}$$

The AC susceptibility $\chi(\omega)$ is determined from $g(\tau)$, as follows:

$$\hat{\chi}(\omega) \equiv \frac{\chi(\omega)}{N} = \frac{1}{T}\int_0^{\infty} \mathrm{d}\tau\, \frac{g(\tau)}{1 - \mathrm{i}\omega\tau}. \tag{7}$$

The validity of assumption (3) will be discussed in section 4, compared with Sibani's (1987) assumption.

3. Results

In this section, we first give numerical results for $\hat{\chi}(\omega)$ and the relaxation time distribution $\hat{g}(\log_{10}\tau)$ on a logarithmic scale (section 3.1). We consider three cases: case I, $\Delta_m =$

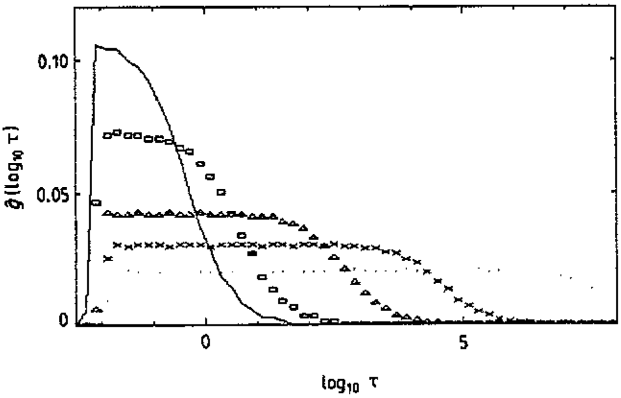

Figure 2. $\hat{g}(\log_{10}\tau)$ versus $\log_{10}\tau$ for case I at $T/J = 2.3$ (———), 1.9 (□), 1.5 (△), 1.3 (×) and 1.1 (. . .) with $\Delta/J = 0.014$ and $n = 1400J/(T - T_c)$ where $T_c/J = 0.3$.

$m\Delta$; case II, $\Delta_m = \Delta \ln m$; case III, $\Delta_m = m^\alpha\Delta$, $0 < \alpha < 1$. In case I, $\hat{\chi}(\omega)$ and the slow part of $\hat{g}(\log_{10}\tau)$ in the 3D $\pm J$ Ising SG can be reproduced. In case III with $\alpha \simeq 0.5$, the values for the 2D $\pm J$ Ising SG can be reproduced. The results for case II resemble those obtained by Sibani (1987). In each case, analytic results for (6) and (7) are also discussed (section 3.2).

3.1. Numerical results

In the numerical calculation of $\hat{\chi}(\omega)$, we fix the frequencies at $\omega = 2\pi/20$, $2\pi/200$ and $2\pi/2000$, and we choose $n = 1000$–2000. In the above three cases (I, II and III), Δ/J was chosen such that the temperatures at which Re $\hat{\chi}(\omega)$ exhibits a peak in the range $1 < T/J < 2$.

3.1.1. Case I: $\Delta_m = m\Delta$. $\hat{\chi}(\omega)$-values for $\Delta/J = 0.014$ are shown for $n = 1000$ and $n = 1400J/(T - T_c)$ where $T_c/J = 0.3$ in figures 1(*a*) and 1(*b*), respectively. $\hat{\chi}(\omega)$ in figure 1(*b*) resembles the $\hat{\chi}(\omega)$ obtained by SSI in the 3D Ising SG at the following points: at low temperatures, Im $\hat{\chi}(\omega)$ exhibits no frequency dependence and is proportional to $T - T_c$ where $T_c \simeq 0.2J$ which is estimated from figure 2(*b*) of SSI. $\hat{g}(\log_{10}\tau)$-values for $n = 1400J/(T - T_c)$ with $T_c/J = 0.3$ are shown in figure 2 at several temperatures. These $\hat{g}(\log_{10}\tau)$-values are also similar to those for the 3D Ising SG, except for the fast parts of $\hat{g}(\log_{10}\tau)$.

3.1.2. Case II: $\Delta_m = \Delta \ln m$. $\hat{\chi}(\omega)$-values for $\Delta/J = 1.5$ are shown for $n = 1000$ and $n = 1000J/T$ in figures 3(*a*) and 3(*b*), respectively. The difference between $\hat{\chi}(\omega)$ for $n = 1000$ and $n = 1000J/T$ is small in this case. We see that the magnitude of [Im $\hat{\chi}(\omega)$]/[Re $\hat{\chi}(\omega)$] is larger than the experimental values (see, e.g., Dekker *et al* 1989) and the behaviour of $\hat{\chi}(\omega)$ resembles that obtained by Sibani (1987). $\hat{g}(\log_{10}\tau)$ versus $\log_{10}\tau$ for $n = 1000J/T$ is shown in figure 4. The peak of $\hat{g}(\log_{10}\tau)$ is too sharp, compared with those of the slow parts of $\hat{g}(\log_{10}\tau)$ for the 2D Ising SG (see figure 6(*a*) of SSI).

3.1.3. Case III: $\Delta_m = m^\alpha\Delta$, $0 < \alpha < 1$. Here we show numerical results only for $\alpha = 0.5$. $\hat{\chi}(\omega)$-values for $\Delta/J = 0.45$ are shown for $n = 1000$ and $n = 1200J/T$ in figures 5(*a*) and 5(*b*), respectively. Good similarities between this case and the 2D Ising SG can be seen

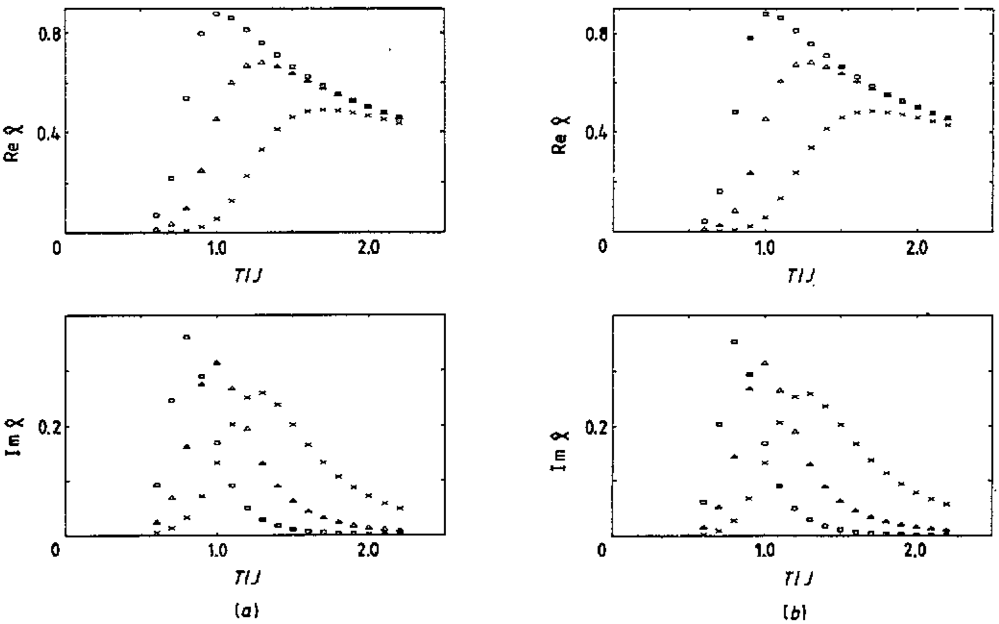

Figure 3. Re $\hat{\chi}$ and Im $\hat{\chi}$ versus T/J for case II with $\Delta/J = 1.5$, $\omega = 2\pi/20$ ($\times$), $2\pi/200$ ($\triangle$) and $2\pi/2000$ ($\square$): (a) $n = 1000$; (b) $n = 1000\,J/T$.

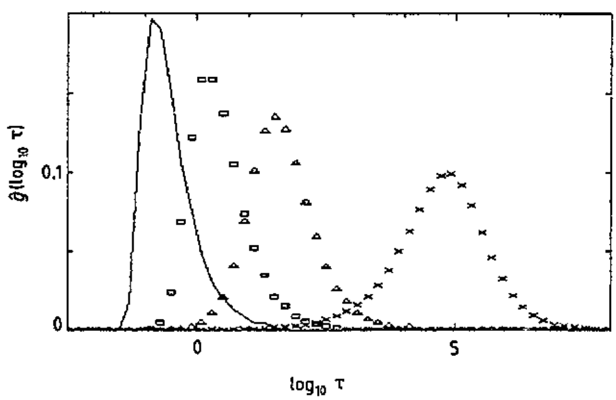

Figure 4. $\hat{g}(\log_{10}\tau)$ versus $\log_{10}\tau$ for case II at $T/J = 2.2$ (——), 1.4 ($\square$), 1.0 ($\triangle$) and 0.6 ($\times$) with $\Delta/J = 1.5$ and $n = 1000\,J/T$.

for the magnitude of [Im $\hat{\chi}(\omega)$]/[Re $\hat{\chi}(\omega)$] and the frequency dependence of the peak value of Im $\hat{\chi}(\omega)$. The temperature dependence of n will be discussed in detail in section 3.2.

Figure 6 shows $\hat{g}(\log_{10}\tau)$ versus $\log_{10}\tau$ at several temperatures when $n = 1200\,J/T$. These $\hat{g}(\log_{10}\tau)$-values are markedly similar to the slow parts of $\hat{g}(\log_{10}\tau)$ for the 2D Ising SG (see figure 6(a) of SSI).

3.2. *Analytical results*

First we calculate $\hat{\chi}(\omega)$ analytically from (7) for $\lambda_n \ll \omega \ll \lambda_1$ and examine whether or not the LSB relation holds for each of the cases I, II and III. Then we discuss long-time

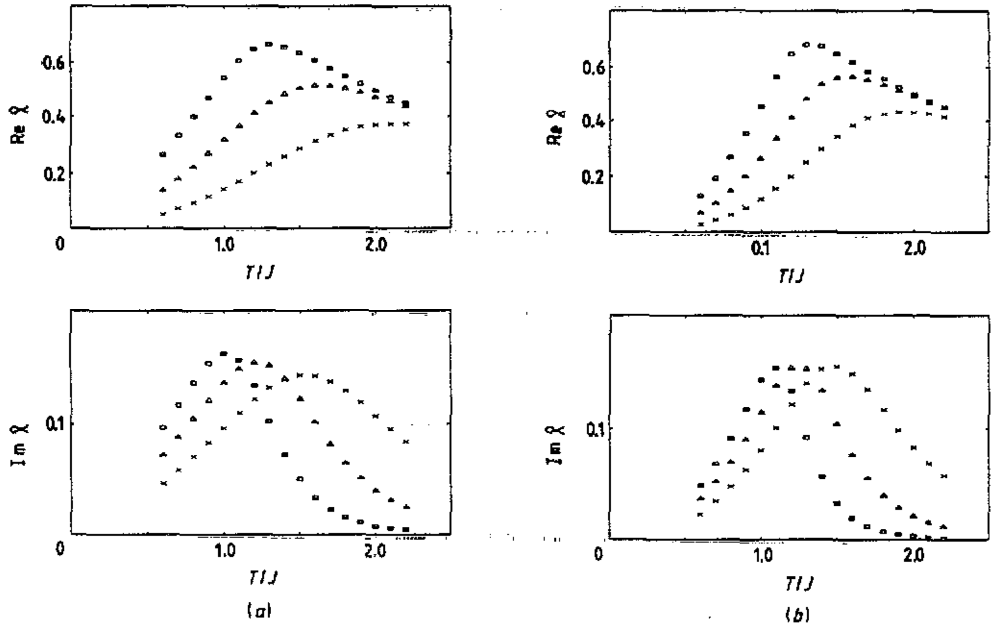

Figure 5. Re $\hat{\chi}$ and Im $\hat{\chi}$ versus T/J for case III with $\Delta/J = 0.45$, $\alpha = 0.5$, $\omega = 2\pi/20$ (×), $2\pi/200$ (△) and $2\pi/2000$ (□): (*a*) $n = 1000$; (*b*) $n = 1200J/T$.

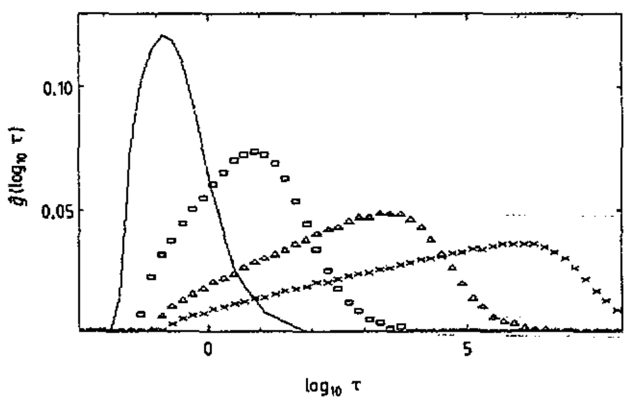

Figure 6. $\hat{g}(\log_{10}\tau)$ versus $\log_{10}\tau$ for case III at $T/J = 2.2$ (——), 1.4 (□), 1.0 (△) and 0.8 (×) with $\Delta/J = 0.45$, $\alpha = 0.5$ and $n = 1200\,J/T$.

behaviour of $C(t)$ for $\lambda_n^{-1} \gg t \gg \lambda_1^{-1}$ using the fluctuation dissipation theorem (FDT), $C(\omega) = (2T/\omega)\,\mathrm{Im}\,\chi(\omega)$. We also study the temperature dependence of the average relaxation time $\tau_{\mathrm{av}} \equiv \int_0^\infty \mathrm{d}\tau\, \tau g(\tau)$.

From (6) and (7), $\hat{\chi}(\omega)$ and τ_{av} are expressed by

$$\hat{\chi}(\omega) = \frac{1}{nT}\sum_{m=1}^{n}\frac{\lambda_m}{\lambda_m - \mathrm{i}\omega} \qquad (8)$$

$$\tau_{\mathrm{av}} = \frac{1}{n}\sum_{m=1}^{n}\frac{1}{\lambda_m} \qquad (9)$$

where we assume that n is an integer so large that the factor $1-2^{-n}$ in (6) can be neglected, and the 2^{-m} terms can also be neglected, unless only first a few terms in the sum make dominant contributions. The condition that the 2^{-m} terms can be neglected in the calculation of $\hat{\chi}(\omega)$ is $\lambda_1 > \lambda_2 > \ldots \gg \omega$, and hereafter we concentrate on only the case when $\lambda_1 > \lambda_2 > \ldots \gg \omega \gg \ldots > \lambda_{n-1} > \lambda_n$.

3.2.1. Case I: $\Delta_m = m\Delta$. From (2), λ_m is given by

$$\lambda_m = (2R^m - R^{m+1} - R^{n+1})/(1-R) \qquad R \equiv \exp(-\Delta/T). \tag{10}$$

Substituting (10) into (8) and converting the sum to an integral ($\Sigma_{m=1}^{n} \ldots \rightarrow \int_1^n \mathrm{d}x \ldots$), $\hat{\chi}(\omega)$ is given by

$$\hat{\chi}(\omega) \simeq \frac{1}{Tn \ln R} \int_{R(2-R)}^{R^n(2-R)} \mathrm{d}z \frac{1 - R^{n+1}/z}{z - R^{n+1} - \mathrm{i}\omega(1-R)}$$

using an integration variable $z = (2-R)R^x$. The R^{n+1}/z term in the integrand can be neglected for $\lambda_1 \gg \omega \gg \lambda_n$, and finally we obtain

$$\begin{aligned}\hat{\chi}(\omega) &\simeq (1/n\Delta)\{\ln[R(2-R)/(1-R)] - \ln(-\mathrm{i}\omega)\}\\ &= (1/n\Delta)\{\ln[R(2-R)/(1-R)] - \ln\omega + \mathrm{i}(\pi/2)\}.\end{aligned} \tag{11}$$

This result reveals two important properties.

(*a*) The LSB relation holds exactly for $\lambda_1 \gg \omega \gg \lambda_n$.

(*b*) $\mathrm{Im}\,\hat{\chi}(\omega)$ exhibits no frequency dependence for $\lambda_1 \gg \omega \gg \lambda_n$.

From (*b*) and the FDT $C(\omega) = (2T/\omega)\,\mathrm{Im}\,\hat{\chi}(\omega)$, we see that the fluctuation spectrum of the magnetization exhibits $1/f$-noise behaviour, and the long-time behaviour of $C(t)$ is the $\ln t$ relaxation because $C(t) \simeq \int^{1/t} \mathrm{d}\omega\, C(\omega) \simeq C_1 - C_2 \ln t$. The condition $\omega \gg \lambda_n$ is satisfied in the low temperature, and (11) almost coincides with the numerical results in section 3.1 below the temperature at which $\mathrm{Im}\,\hat{\chi}(\omega)$ exhibits a peak.

By performing a similar calculation, τ_{av} is given by

$$\tau_{\mathrm{av}} \simeq [T(1-R)/n\Delta R] \ln[(2-R)/(2-2R)] \exp(n\Delta/T). \tag{12}$$

If we assume that $n = n_0 J/(T - T_c)$, where n_0 is a constant, for which $\hat{\chi}(\omega)$ in the 3D Ising SG can be reproduced more accurately, we obtain a Vogel–Fulcher-like form $\tau_{\mathrm{av}} \propto \exp[n_0 J\Delta/T(T-T_c)]$.

3.2.2. Case II: $\Delta_m = \Delta \ln m$. This case corresponds to neither the 3D Ising SG nor the 2D Ising SG, and so we simply describe only the results. As pointed out by OS, when $\Delta/T \leq 1$, $\langle d(t)\rangle$ does not converge in the limit $n \rightarrow \infty$, and therefore we assume that $\Delta/T > 1$. For frequencies in the range $\lambda_1 \gg \omega \gg \lambda_n$, it is shown that $\mathrm{Im}\,\hat{\chi}(\omega) \propto 1/\omega$, and the LSB relation does not hold, as easily expected from the narrowness of the peak width of $\hat{g}(\log_{10}\tau)$ as shown in figure 4 (Lundgren *et al* 1981). The long-time relaxation of $C(t)$ is proportional to t in the narrow range $\lambda_n^{-1} \gg t \gg \lambda_1^{-1}$.

3.2.3. Case III: $\Delta_m = m^\alpha\Delta$, $0 < \alpha < 1$. First we give an approximate expression for λ_m to estimate the sums of (8) and (9). Substituting $a_m = R^{m^\alpha}$ into (2) and converting the sum to an integral, we obtain

$$\lambda_m \simeq R^{m^\alpha} + \int_m^n \mathrm{d}x\, R^{x^\alpha}. \tag{13}$$

Replacing the variable x by $z = x^\alpha \ln R$, we estimate the integral in (13) as

$$\begin{aligned}\int_m^n \mathrm{d}x\, R^{x^\alpha} &= \int_{m^\alpha \ln R}^{n^\alpha \ln R} \mathrm{d}z \exp z \frac{z^{(1-\alpha)/\alpha}}{\alpha(\ln R)^{1/\alpha}}\\ &\simeq b_1 m^{1-\alpha} R^{m^\alpha} - b_1 n^{1-\alpha} R^{n^\alpha} + \mathrm{O}(m^{1-2\alpha} R^{m^\alpha})\end{aligned} \tag{14}$$

where $b_1 \equiv -1/(\alpha \ln R) = T/\alpha\Delta$ and the last equality in (14) is derived by partial integration. When $n \gg m \gg 1$, λ_m is given by

$$\lambda_m \simeq b_1 m^{1-\alpha} R^{m^\alpha}. \tag{15}$$

When $\lambda_1 \gg \omega \gg \lambda_n$ and α is not too small, the main contributions in (8) and (9) will come from summations in the range $m, n-m \gg 1$. Therefore we approximately use (15) for λ_m in the following calculations.

We first calculate the AC susceptibility. Substituting (15) into (8) and converting the sum to an integral, we obtain

$$\hat{\chi}(\omega) \simeq \frac{1}{Tn} \int_1^n \mathrm{d}x \frac{x^{1-\alpha} R^{x^\alpha}}{x^{1-\alpha} R^{x^\alpha} - \mathrm{i}\omega/b_1}. \tag{16}$$

As we could not evaluate the integral

$$f(a) \equiv \int_1^n \mathrm{d}x \frac{x^{1-\alpha} R^{x^\alpha}}{x^{1-\alpha} R^{x^\alpha} + a}$$

analytically, we estimate it numerically for $R \gg a \gg n^{1-\alpha} R^{n^\alpha}$. We obtained, for example, $f(a) = -\bar{c}_1 \ln a + \bar{c}_2 (\ln a)^2$ for $\alpha = \frac{1}{2}$ and $f(a) = -\tilde{c}_1 \ln a + \tilde{c}_2 (\ln a)^2 - \tilde{c}_3 (\ln a)^3$ for $\alpha = \frac{1}{3}$. From these results we expect that $f(a)$ is well described in a form of kth polynomials of $\ln a$ for $\alpha = 1/k$. Hereafter we assume that this expectation is correct.

Comparing these results with those of the MC simulation by SSI, we find that $\hat{\chi}(\omega)$ in the 2D Ising SG can be better reproduced for $\alpha = \frac{1}{2}$ for the following two reasons.

(i) At low temperatures $\mathrm{Im}\,\hat{\chi}(\omega)$ obtained by the MC simulation can be roughly regarded as $\mathrm{Im}\,\hat{\chi}(\omega) = c_1 - c_2 \ln \omega$ ($c_1, c_2 > 0$). This corresponds to the case when $\alpha = \frac{1}{2}$ in (16).

(ii) In the MC simulation, the LSB relation well holds in the 2D Ising SG as well as in the 3D Ising SG (Suzuki *et al* 1991). There are only two cases, $\alpha = 1$ and $\alpha = \frac{1}{2}$, where the LSB relation exactly holds in our calculation (the terms, proportional to $[\ln(-\mathrm{i}\omega)]^3$, $[\ln(-\mathrm{i}\omega)]^4, \ldots$, which violate the LSB relation appear for $\alpha = 1/k, k \geq 3$). This also supports $\alpha \simeq \frac{1}{2}$.

Therefore, assuming that $\alpha = \frac{1}{2}$, we consider the temperature dependence of n in the 2D Ising SG, comparing the above results with those of SSI. From the least-squares fitting for R-values in the range $0.65 > R > 0.4$, we obtained

$$f(a) = -4.55\,(-1/\ln R)^{2.72} \ln a + 1.196(-1/\ln R)^{1.785} (\ln a)^2.$$

This suggests a functional form, $f(a) = -\hat{c}_1 T^3 \ln a + \hat{c}_2 T^2 (\ln a)^2$, where $\hat{c}_1$ and $\hat{c}_2$ are positive constants. Then, assuming this and $n = n_0/T^\delta$, we obtain from (16)

$$\mathrm{Im}\,\hat{\chi}(\omega) = (T^{\delta-1}/n_0)\,\mathrm{Im} f(-\mathrm{i}\omega/b_1) = (\pi T^{\delta-1}/2n_0)[\hat{c}_1 T^3 - 2\hat{c}_2 T^2 \ln(\omega/b_1)]. \tag{17}$$

From (17) the coefficient of $\ln \omega$ in $\mathrm{Im}\,\hat{\chi}(\omega)$ is proportional to $T^{1+\delta}$. We estimate that $0 \leq \delta < 1$ from the temperature dependence of the coefficient of $\ln \omega$ in $\mathrm{Im}\,\hat{\chi}(\omega)$ obtained by SSI in the 2D Ising SG, although it is difficult to determine δ accurately owing to large scatter in the data.

As for the long-time relaxation of $C(t)$, it is shown from $C(t) = \int^{1/t} \mathrm{d}\omega (2T/\omega)\,\mathrm{Im}\,\chi(\omega)$ and $\mathrm{Im}\,\hat{\chi}(\omega) = c_1 - c_2 \ln \omega$ that $C(t)/N \simeq c_0 - 2Tc_1 \ln t - Tc_2 (\ln t)^2$.

Finally we evaluate the average relaxation time τ_{av}. Substituting (15) into (9) and converting the sum to an integral, we obtain

$$\tau_{\mathrm{av}} \simeq \frac{1}{nb_1} \int_1^n \mathrm{d}x\, x^{\alpha-1} \exp\left(\frac{x^\alpha \Delta}{T}\right) = \frac{1}{n}\left[\exp\left(\frac{n^\alpha \Delta}{T}\right) - \exp\left(\frac{\Delta}{T}\right)\right]. \tag{18}$$

If we assume that $n = n_0/T^\delta$, τ_{av} follows the generalized Arrhenius law

$\tau_{av} \propto T^{\delta} \exp(n_0^{\sigma}\Delta/T^{\sigma})$, where $\sigma = 1 + \delta\alpha$. From the above results ($\alpha = \frac{1}{2}$ and $0 \leqslant \delta < 1$), we obtain $1 \leqslant \sigma < 1.5$. This value of σ is consistent with that obtained by SSI for the 2D Ising SG.

4. Discussion

First we compare our results with those of Sibani and discuss the origin of the difference between them. On the basis of an assumption different from (3), Sibani (1987) obtained a correlation function $C(t)$ and AC susceptibilities similar to those for case II. Further he showed that the LSB relation holds only in the limit $z \rightarrow 1$. On the other hand, we show in this paper that $\hat{\chi}(\omega)$ in the 2D and 3D $\pm J$ Ising SGs can be reproduced in the OS model on the basis of assumption (3). It is shown that the difference between the 3D and 2D systems corresponds to a difference in the scaling of free-energy barriers with ultrametric distance, i.e. a difference between $k = 1$ and 2 in $\Delta_m = m^{1/k}\Delta$, and that the LSB relation holds exactly for both cases $k = 1$ and $k = 2$ with a large enough number of levels $n \gg 1$ and $\lambda_1 \gg \omega \gg \lambda_n$.

Thus our results are a better approximation for the real experimental results than Sibani's results. We think that these differences come from the use of different assumptions. These assumptions are roughly expressed as follows: our assumption is

$$\frac{C(t)}{N} \simeq \sum_{k=0}^{2^n-1} \left(1 - \frac{d(k,0)}{n}\right) P_k(t)$$

and Sibani's assumption is

$$\frac{C(t)}{N} \simeq \sum_{k=0}^{2^n-1} \left(1 - \frac{z^{d(k,0)}}{z^n}\right) P_k(t)$$

where $z > 1$. In Sibani's expression for $C(t)$, only the terms with a few $d(k, 0)$ close to n contribute relevantly to the relaxation of $C(t)$, except for the limit $z \rightarrow 1$. This makes the density of the slowest part of $\hat{g}(\log_{10}\tau)$ considerably larger than our value. Therefore it is supposed that in Sibani's assumption a peak width of $\hat{g}(\log_{10}\tau)$ large enough to reproduce the experimental AC susceptibilities could not be made even at low temperatures.

In this paper, we express the correlation function $C(t)$ by means of the average ultrametric distance $\langle d(t)\rangle$ using (3) and get good agreement between the present results and those in the previous MC simulation. There remains, however, one difficulty to be overcome in order to reduce the dynamics in the 3D $\pm J$ Ising SG to the dynamics in case I. This is the temperature of the phase transition and the behaviour of the temperature dependence of τ_{av}. At present as the result of large-scale simulations in both space and time (Ogielski 1985, Bhatt and Young 1985), it is believed that the phase transition temperature T_g in the 3D $\pm J$ Ising SG is $T_g/J = 1.175 \pm 0.025$ and τ_{av} exhibits a divergent behaviour near T_g according to the power law. On the other hand, if we assume that $n \propto 1/(T - T_c)$ in case I which reproduces the best $\hat{\chi}(\omega)$ for the 3D Ising SG, the phase transition temperature could be regarded as $T_c(\simeq 0.2J)$ and τ_{av} exhibits a divergent behaviour near T_c according to a Vogel–Fulcher-like law.

In order to overcome this difficulty, there are several possibilities to be considered.

(1) There is the possibility that the dynamics on a time scale investigated for $\hat{\chi}(\omega)$ by SSI could be essentially different from those on a long-time scale near and below the critical temperature.

(2) The model of the dynamics in an ultrametric space used in this paper is the simplest. There might be the possibility that another more complicated model of dynamics in an ultrametric space could overcome this difficulty.

Because of possibility (2), the characteristic properties in a variety of models of dynamics in an ultrametric space should be studied. In order to investigate possibility (1), AC susceptibilities with longer cycles should be studied by the MC simulation in larger systems. If possibility (1) were true, we might observe a crossover behaviour for small frequencies from $\mathrm{Im}\,\chi(\omega) = $ constant found by SSI to $\mathrm{Im}\,\chi(\omega) \propto |\ln\omega|^{-(1+\theta/\varphi)}$ predicted by Fisher and Huse (1986, 1988), based on an *ansatz* for the scaling of low-lying large-scale droplet excitations.

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References

Bachas C P and Huberman B A 1987 *J. Phys. A: Math. Gen.* **20** 4995
Bhatt R N and Young A P 1985 *Phys. Rev. Lett.* **54** 924
Dekker C, Arts A F M, de Wijn H W, van Duyneveldt A J and Mydosh J A 1989 *Phys. Rev.* B **40** 11243
Fisher D S and Huse D A 1986 *Phys. Rev. Lett.* **56** 1601
—— 1988 *Phys. Rev.* B **38** 386
Gunnarsson K, Svedlindh P, Nordblad P, Lundgren L, Aruga H and Ito A 1988 *Phys. Rev. Lett.* **61** 754
Lundgren L, Svedlindh P and Beckman O 1981 *J. Magn. Magn. Mater.* **25** 33
Mezard M, Parisi G, Sourlas N, Toulouse G and Virasoro M 1984 *J. Physique* **45** 843
Ogielski A T 1985 *Phys. Rev.* B **32** 7384
Ogielski A T and Stein D L 1985 *Phys. Rev. Lett.* **55** 1634
Parisi G 1980 *J. Phys. A: Math. Gen.* **13** L115, 1101, 1887
Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1792
Sibani P 1987 *Phys. Rev.* B **35** 8572
Suzuki M, Shirakura T and Inawashiro S 1991 *J. Phys.: Condens. Matter* **3** 3139